Math 434 Assignment 2

Due March 29

Assignments will be collected in class.

1. Let α be an ordinal and $X \subseteq \alpha$. Prove that $(X, \in)$ is a well-order and that $ot(X, \in) \leq \alpha$.

2. Prove that ordinal addition and multiplication are associative ($(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ and similarly for multiplication).

3. Let α and β be ordinals with $\alpha < \beta$. Prove that there is a unique ordinal δ such that $\alpha + \delta = \beta$.

4. Prove Zorn's Lemma. *Hint: Look at the proof that every vector space has a basis.*

5. Use Zorn's Lemma to show that every consistent theory has a complete consistent extension. That is, let T be a consistent $\mathcal{L}$-theory; show that there is a complete consistent $\mathcal{L}$-theory $T' \supseteq T$.

6. A real number is algebraic if it satisfies a polynomial equation with coeffecients in $\mathbb{N}$. Show that there are countably many algebraic numbers.

For the next few problems, we will work in the topological space $\mathbb{R}$ though the results hold more generally. Recall that a x is a *limit point* of X if $x \in \overline{X - \{x\}}$, and that a set is closed if and only if it contains all of its limit points. A point $x \in X$ is an *isolated point* of X if it is not a limit point of X, or equivalently, there is a neighbourhood of x that does not intersect X. A set is *perfect* if it is closed and has no isolated points.

7. Show that any collection of disjoint open sets in $\mathbb{R}$ is countable.

8. We will show that any non-empty perfect set $X \subseteq \mathbb{R}$ has cardinality $2^{\aleph_0} = |\mathbb{R}|$.

   (a) Let $2^{<\omega}$ be the set of finite sequence of 0's and 1's, and let 2^{ω} be the set of infinite sequences of 0's and 1's. Let 2^n be the set of length n sequences of 0's and 1's. We denote by $|\sigma|$ the length of σ.

   Construct, for each $\sigma \in 2^{<\omega}$ an open interval U_σ such that:

   - $U_\sigma \cap X \neq \varnothing$ for each σ.
   - Whenever $|\sigma| = |\tau|$, $U_\sigma \cap U_\tau = \varnothing$.
   - Whenever $\sigma \prec \tau$ (τ is a proper extension of σ), $\overline{U_\tau} \subseteq U_\sigma$.

- The diameter of U_σ is at most $2^{-|\sigma|}$.

(b) Show that for each $\pi \in 2^\omega$, there is a point $x_\pi \in \bigcap_{\sigma<\pi} U_\sigma$, and that $x_\pi \neq x_{\pi'}$ for $\pi \neq \pi'$.

(c) Conclude that $|X| = |2^\omega| = 2^{\aleph_0}$.

9. Given a closed set $X \subseteq \mathbb{R}$, the Cantor-Bendixson derivative of X is the set X' of all limit points of X. Since X is closed, $X' \subseteq X$. Define:

- $X^0 = X$,
- $X^{\alpha+1} = (X^\alpha)'$,
- $X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$ for λ a limit ordinal.

(a) Prove by transfinite induction that each X^α is closed.

(b) Prove that for every closed set $X \subseteq \mathbb{R}$, $X - X'$ is countable.

(c) Prove that the sequence (X_α) is eventually constant, and that this happens at a countable ordinal: For any closed set X, there is a countable ordinal α such that for all $\beta \geq \alpha$, $X^\alpha = X^\beta$. *Hint: If $X \neq X'$, then there is an interval with rational enpoints that intersects X but not X'.*

(d) Show that X^α is perfect, where α is as above.

(e) Prove that every closed subset of $\mathbb{R}$ is a union of a perfect set and a countable set, either of which might be empty.

In particular, every closed set of $\mathbb{R}$ has cardinality $\leq \aleph_0$ or cardinality $2^{\aleph_0}$, that is, the continuum hypothesis is true for closed subsets of $\mathbb{R}$.